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Gauge Kinematics of Deformable Bodies

Alfred Shapere and Frank Wilczek

Author:
Ankit Mahajan
Roll No. 120100027

Supervisor:
Prof. Ravi Banavar

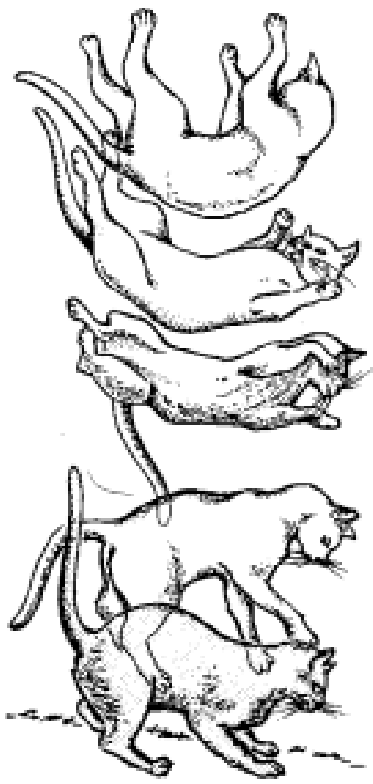

Figure 1: Falling cat

Introduction

A deformable body with zero angular momentum can change orientation by undergoing a sequence of deformations. For example, a cat dropped upside-down manages to land on its feet without any external torques (in COM frame). It does this by undergoing shape changes. In the paper, this rotation is naturally expressed in a geometric form - a gauge potential (connection) over the configuration space. This change in orientation is shown to be a manifestation of conservation of angular momentum.

Gauge theories form a corner stone of theoretical particle physics (standard model). When applied to processes at subatomic level, gauge theories are rather abstract and non-intuitive. But the same framework can be used to describe the motion of deformable bodies, which is much easier to visualize, and thus provides a clearer view of the concepts involved.

The Setting

Let's first formulate the problem in concrete mathematical terms. Consider a deformable body of fixed mass. For the sake of simplicity let's ignore translation (equivalent to working in COM frame in absence of external forces). We have two relevant spaces: the space of unoriented shapes and the space of oriented shapes. The former can be thought of as the latter modulo rotations about COM. If the body undergoes a sequence of deformations, it will change orientation so as to conserve angular momentum. It can be shown that this change in orientation is independent of the time *rate of change of shape*. Even though this appears straightforward at first glance, there is a subtle problem underneath the simple appearance. How can we compare orientations of different oriented shapes? Mathematical structures called *principal bundles* and *connections*, known as gauge potentials, on these structures provide a natural setting for addressing this problem. In the following sections, these mathematical concepts are briefly reviewed. It will be shown that the problem of dynamics of deformable bodies can be cast and solved in terms of these structures.

Principal Fiber Bundles

Definition: A C^∞ principal fiber bundle consists of a manifold Q (called the total space), a Lie group G, a base manifold M, a smooth right action $\sigma : Q \times G \to Q$ and a smooth projection map $\pi : Q \to M$ such that the following conditions are true:

- σ preserves the fibers of π,

  $$\pi(q.g) = \pi(q)$$

  for all $q \in Q$ and $g \in G$

- For each $x \in M$ there exists an open neighborhood V of x in M and a homeomorphism $\Psi : \pi^{-1}(V) \to V \times G$ of the form,

  $$\Psi(q) = (\pi(q), \psi(q))$$

  where $\psi : \pi^{-1}(V) \to G$ satisfies

  $$\psi(q.g) = \psi(q)g$$

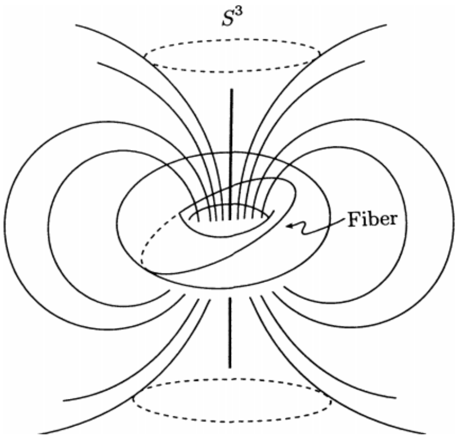

Figure 2: Hopf bundle[5]

Q is said to be a principal G-bundle over M and this is indicated diagrammatically by writing $G \hookrightarrow Q \to M$.

A principal G-bundle is called trivial if it is globally isomorphic to $M \times G$. However, in general principal bundles are only locally trivial.

For each $x \in M$, $\pi^{-1}(x)$ is a closed submanifold of Q, called the fiber over x. If u is a point of $\pi^{-1}(x)$, then $\pi^{-1}(x)$ is the set of points $u.a$, $a \in G$ and is called the fiber through u. Every fiber is diffeomorphic to G. Each fiber can be thought of as a copy of G without a specified identity element (such structures are called torsors, a notion similar to affine spaces). M is the quotient space of Q by the equivalence relation induced by G, $M = Q/G$.

Example of a nontrivial principal bundle: The group $SU(2)$ can be described as a $U(1)$-bundle over S^2. This is the famous Hopf bundle (2).

Configuration space (oriented shapes) of a deformable body in three dimensions is a principal fiber bundle with M as the shape space and G as $SO(3)$. In this case the definitions of π and σ are obvious.

Choice of Gauge

Equipped with the formalism of principal bundles, we can now appropriately formulate the problem of dynamics of deformable bodies. We need to find a way to compare orientations of different shapes. This can be achieved by assigning a standard orientation, a set of body fixed axes, to all shapes. All other orientations can be measured with respect to the standard one. Unfortunately, there may not exist a canonical choice of standard orientations, in general. It should be noted that, as long as it is sufficiently smooth, the particular choice of standard orientations does not affect the final result. So we are at a liberty make a convenient choice of an orientation, in a smooth manner, over the space of unoriented shapes. This is known as the choice of gauge (also known as a cross section). Another subtlety needs to addressed here - does such a smooth choice always exist? It turns out, in fact, a global choice of gauge cannot be made for nontrivial bundles.

Theorem: A principal G-bundle $Q : Q \to M$ is trivial iff it admits a global cross-section $s : M \to Q$.

Thus in general a choice of gauge can only be made locally. It might be tempting, at this point, to say that the space of oriented shapes is isomorphic to the Cartesian product of the space of unoriented shapes and the rotation group, so why bother with the principal bundle? To answer this, note that this isomorphism is not canonical as explained before. In modeling the configuration space as a trivial bundle one has to relinquish the important gauge symmetry inherent to the problem. As we will see in the following sections, the principal bundle formalism exploits this gauge invariance symmetry to produce an elegant geometric theory of dynamics.

Connections on principal bundles

We would like to *separate* the infinitesimal orientation change caused by shape change from that caused by rigid rotation. Gauge potentials or connections afford a way to do just that.

Given any point $q \in Q$, let $\sigma_q(g) = q.g$. Then we have the following composition of maps,

$$G \xrightarrow{\sigma_q} Q \xrightarrow{\pi} M$$

The map σ_q is one-to-one, the map π is onto, and $\pi^{-1}(x) = image(\sigma_q)$ where $\pi(q) = x$. Corresponding to this we have the following sequence of derivative maps,

$$\mathcal{G} \xrightarrow{\sigma_{q*id}} T_qQ \xrightarrow{\pi_*q} T_xM$$

This is an exact sequence of linear maps: σ_{q*id} is one-to-one, π_*q is onto and $ker(\pi_*q) = image(\sigma_{q*id})$.

Definition: The image of σ_{q*id} is called the "vertical subspace" V_q at q.

Vectors in V_q are tangent to the fiber through q. We have $V_q = T_q(\pi^{-1}(x))$. Note that σ_{q*id} provides a canonical isomorphism between $\mathcal{G}$ and V_q.

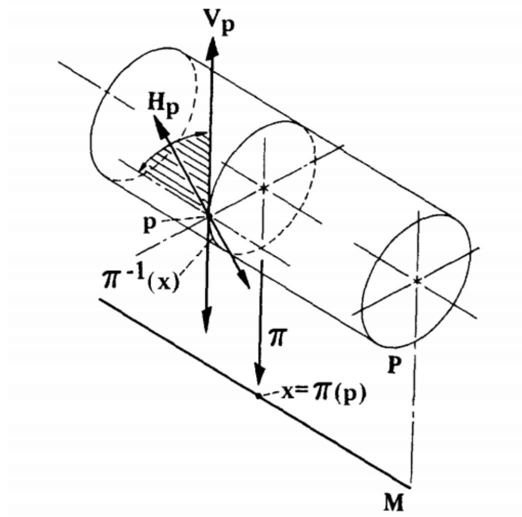

Figure 3: Vertical and Horizontal subspaces[1]

In the case of a deformable body in three-space, $G = SO(3)$, and $\mathcal{G} = \mathfrak{so}(3)$. Thus $\mathcal{G}$ can be identified with $\mathbf{R}^3$. A vector $\mathbf{A} \in \mathcal{G}$ represents an instantaneous angular velocity. The infinitesimal action σ_{q*id} is the infinitesimal rotation of the configuration q about the axis $\mathbf{A}$.

$$\sigma_{q*id}(\mathbf{A})(y) = \mathbf{A} \times \mathbf{q}(y)$$

Here the y's denote the points of the deformable body, so that $q(y) \in \mathbf{R}^3$ is the initial position of the body point labeled y when the body is in the configuration q. V_q represents the space of all rigid transformations of q.

We now give two equivalent definitions of a connection on Q.

Definition 1:A horizontal distribution is a smoothly varying family

$$H_q \in T_q Q$$

of linear subspaces complementary to the vertical distribution and invariant under the G action. Thus

$$T_q Q = H_q \oplus V_q$$

and

$$\sigma_{g*}(H_q) = H_{q.g}$$

Definition 2: A $\mathcal{G}$-valued connection one-form is a smoothly varying family

$$\omega_q : T_qQ \to \mathcal{G}$$

such that,

$$\omega_q(\sigma_{q*id}(\mathbf{A})) = \mathbf{A}$$

$$(\sigma_g)^*\omega = ad_{g^{-1}} \circ \omega$$

Note that, $H_q = ker\ \omega_q$.

Mechanical connection[4]

Suppose Q is a Riemannian manifold and that G acts on Q by isometries. We define the horizontal distribution to be:

$$H_q = V_{q\perp}$$

the orthogonal complement to the vertical space. Invariance under the action of G follows immediately from the fact that G are isometries. This connection is called the mechanical connection when the metric on Q is the kinetic energy metric, induced by the inner product

$$< \delta q_1, \delta q_2 >= \int < \delta q_1(y), \delta q_2(y) > dm(y)$$

dm is the mass distribution and δq_i for $i = 1, 2$, are two deformations of the body, i.e. $\delta q_i \in T_qQ$.

If $\delta q_2 \in V_q$ then,

$$\delta q_2(y) = \mathbf{A} \times \mathbf{q}(y)$$

$$< \delta q_1, \delta q_2 >=< \mathbf{A}, M(q, \delta q_1) >$$

where,

$$M(q, \delta q_1) = \int q(y) \times \delta q_1(y) > dm(y)$$

is the expression for the total angular momentum associated to the deformation δq_1 of the configuaration q. It follows that δq_1 is horizontal iff $M(q, \delta q_1) = 0$. Thus,

$$H_q = \{\delta q \in T_qQ : M(q, \delta q) = 0\}$$

If the horizontal distribution is defined by the vanishing of the angular momentum M then the connection one-form ω has the same kernel as M. Consequently we must have :

$$\omega_q = R_qM(q, \quad)$$

where R_q is invertible. R can be shown to be the inverse of locked moment of inertia tensor. Thus,

$$\omega = I^{-1}M$$

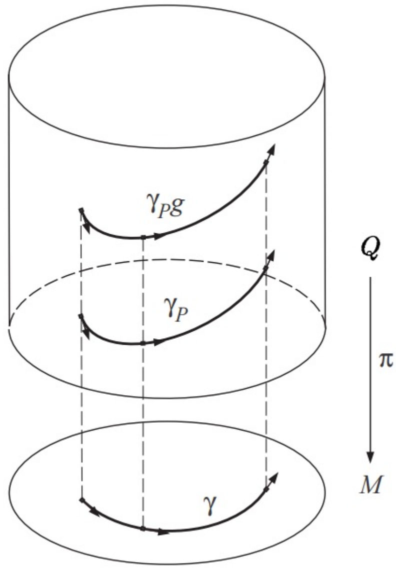

Figure 4: Horizontal lift

Parallel transport

Definition: Let $\gamma : [0,1] \to M$ be a curve in the base manifold (a base curve). A curve $\gamma_Q : [0,1] \to Q$ is called the horizontal lift of γ if

- $\pi(\gamma_Q) = \gamma$
- All tangent vectors X_Q to γ_Q are horizontal: $X_Q \in H_{\gamma_Q}Q$

Theorem: Let $\gamma : [0,1] \to M$ be a base curve and let $q \in \pi^{-1}(\gamma(0))$. Given a connection, there exists a unique horizontal lift γ_Q such that $\gamma_Q(0) = q$.

This means that we can (given a connection) uniquely define the parallel transport of a point $q \in Q$ along a curve γ in M by moving it along the unique horizontal lift of γ through q. We saw in the last section that motions of the deformable body which lie in the horizontal subspace have zero angular momentum. Given a path in the shape space of the deformable body, the unique path horizontally lifted to the configuration space will be its path in real space. Thus the condition of angular momentum conservation can be expressed geometrically in terms of parallel transport on the principal bundle.

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