Nonlinear Dynamics of Hodgkin-Huxley Neurons

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Neurons and Action Potentials

- Neurons: electrically excitable cells that transmit information throughout the body in electrical and chemical signals
- An action potential: an abrupt and transient change of membrane voltage that propagates to other neurons via the axon
- All or none: Only stimuli above a certain "threshold" elicit an action potential response

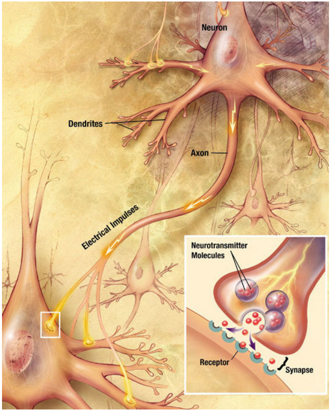


Figure: Communication between neurons (Source: Wikipedia)

Voltage Gated Channels

$$\mathrm{C} \underset{\beta(\mathrm{V})}{\overset{\alpha(\mathrm{V})}{\rightleftharpoons}} \mathrm{O} \tag{1}$$

From the law of mass action,

$$\frac{dm}{dt} = \alpha(V)(1-m) - \beta(V)m = \frac{m_{\infty(V)} - m}{\tau(V)} \tag{2}$$

where,

$$m_{\infty(V)} = \frac{\alpha(V)}{\alpha(V)+\beta(V)}, \qquad \tau(V) = \frac{1}{\alpha(V)+\beta(V)} \tag{3}$$

The rate functions $\alpha(V)$ and $\beta(V)$ are chosen to fit the voltage-clamp experiment data.

Hodgkin Huxley Equations

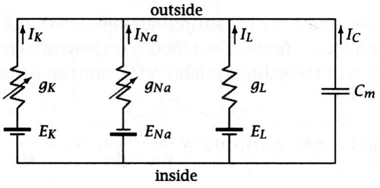


Figure: The equivalent circuit of the squid axon

$$c_m \dot{V} = i - \bar{g}_K n^4 (V - E_K) - \bar{g}_{Na} m^3 h (V - E_{Na}) - g_L (V - E_L) \tag{4a}$$

$$\dot{m} = \alpha_m(V)(1-m) - \beta_m(V) m \tag{4b}$$

$$\dot{n} = \alpha_n(V)(1-n) - \beta_n(V) n \tag{4c}$$

$$\dot{h} = \alpha_h(V)(1-h) - \beta_h(V) h \tag{4d}$$

The V-m reduced system

- This approach, by FitzHugh, although not rigorous presents a vivid picture of the dynamics of the system
- Based on reducing dimensionality by ignoring the dynamics of the variables with large time constants (viz. n and h)

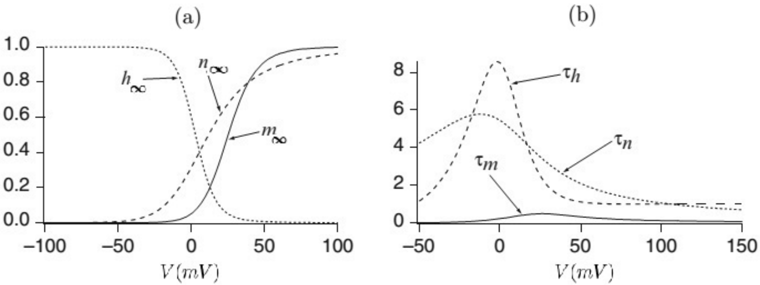


Figure: The steady state values and time constants of gating variables

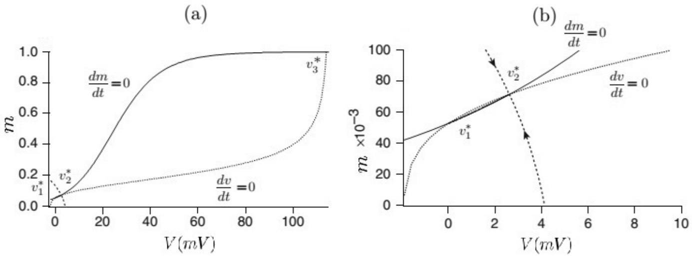


Figure: The V and m nullclines (at rest values of h and n)

Equations of nullclines:

V nullcline: $$m = [\frac{i - \bar{g}_K n^4 (V - E_K) - g_L (V - E_L)}{\bar{g}_{Na} h (V - E_{Na})}]^{1/3} \quad (5)$$

m nullcline: $$m = \frac{\alpha_m(V)}{\alpha_m(V) + \beta_m(V)} = m_\infty(V) \quad (6)$$

Bifurcation Analysis

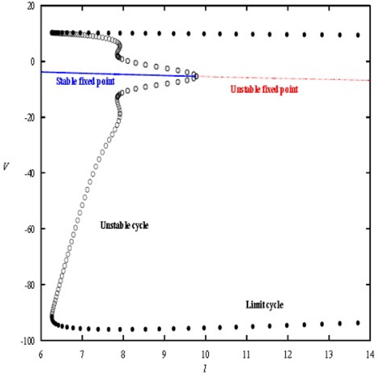
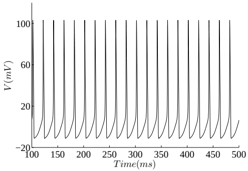


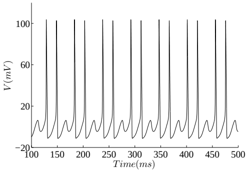
Figure: Bifurcation diagram [S. Lee et al., Physical Review E 73, 041924]

- At $i = 6.3\mu A/cm^2$, a double-cycle bifurcation or saddle-node bifurcation of periodics occurs and a pair of stable and unstable periodic solutions is generated.
- An unstable periodic solution is bifurcated by the sub-critical Hopf bifurcation at $i = 9.8\mu A/cm^2$.

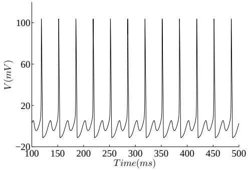
Entrainmment



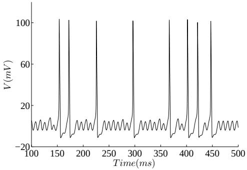
(a) $i = 2\mu A/cm^2, \nu = 50Hz(1:1)$



(b) $i = 2\mu A/cm^2, \nu = 55Hz(2:3)$

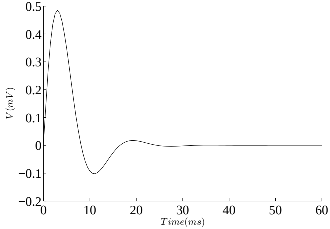


(c) $i = 2\mu A/cm^2, \nu = 60Hz(1:2)$

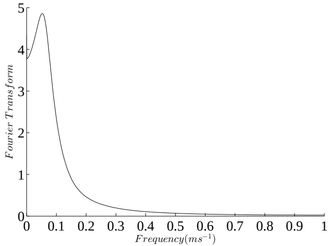


(d) $i = 2.85\mu A/cm^2, \nu = 113Hz(irrational)$

Resonance



(a) V time series



(b) Fourier transform of the V time series

Figure: Unperturbed HH neuron

Note that the fourier transform peaks at around 54 Hz(ν_0).

Resonance

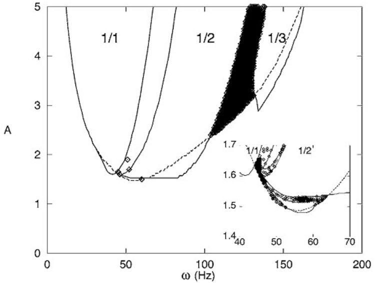


Figure: Arnold Tongue in the parameter space of the forcing amplitude A and frequency ω [S. Lee et al., Physical Review E 73, 041924]

Note that the minimum of the dotted curve denoting the boundary of non-firing region occurs at a frequency approximately equal to ν_0

Active Cable

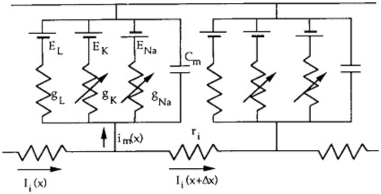


Figure: Equivalent circuit of an active cable

$$c_m \frac{\partial V}{\partial t} = \frac{a}{2r_i}\frac{\partial^2 V}{\partial x^2} - \bar{g}_K n^4 (V - E_K) - \bar{g}_{Na} m^3 h (V - E_{Na}) - g_L (V - E_L) \quad (7)$$

Electromagnetic Perturbation

In the presence of external EM field, the axial voltage matching equation gets modified to,

$$r_i I_i = -\pi a^2 \frac{\partial V}{\partial x} \tag{8}$$

Thus we get,

$$c_m \frac{\partial V}{\partial t} = \frac{a}{2r_i} \frac{\partial^2 V}{\partial x^2} - i_K - i_{Na} - i_L - \frac{a}{2r_i} \frac{\partial E_x}{\partial x} \tag{9}$$

For a solenoidal electromagnetic field, the modified system of equations is numerically integrated and action potential response is observed in a region of parameter space.